Journal of Statistical Physics, Vol. 46, Nos. 3/4, 1987

On The Invariant Measures for the Two-Dimensional Euler Flow

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Received May 2, 1986; revision received November 10, 1986

It is proven that the canonical Gibbs measure associated with a gas of vortices of intensity $\pm\sqrt{\sigma}$ converges, in the limit $N \to \infty$, $\sqrt{\sigma} \to 0$, $N\sigma \to$ const, to a Gaussian measure, which is invariant for the two-dimensional Euler equation.

KEY WORDS: Invariant measures for the Euler flow; vortices; Gaussian random fields.

1. INTRODUCTION

Invariant measures for the two-dimensional Euler flow have been introduced for trying to explain some features of two-dimensional stationary homogeneous turbulence. See Ref. 1 for a physical understanding of the subject and, more generally, for a review on the statistical mechanical approaches to fully developed turbulence.

At a more rigorous level the study of such invariant measures and their connections with analogous problems arising in statistical mechanics and in quantum field theory was started by Hopf[(2)] and has more recently been carried out by several authors.[(3–8)]

We review the basic framework of the above references.

Let us consider the two-dimensional flat torus $T = [-\pi, \pi]^2$ in which an incompressible inviscid fluid is confined. The Euler equation in the vorticity formalism is

$$\partial_t \omega + (u \cdot \nabla)\, \omega = 0 \tag{1.1}$$

$$\operatorname{curl} u = \omega, \qquad \operatorname{div} u = 0 \tag{1.2}$$

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0022-4715/87/0200-0729$05.00/0

where $x \to u(x) \in \mathbb{R}^2$ and $x \to \omega(x) \in \mathbb{R}$ are the velocity and the vorticity field, respectively. Equations (1.2) are equivalent to

$$u(x) = \int_T K(x, y)\, \omega(y)\, dy \tag{1.3}$$

where

$$K(x, y) = \nabla_x^\perp V(x, y), \qquad \nabla^\perp = (\partial_{x_2}, -\partial_{x_1}) \tag{1.4}$$

and

$$V(x, y) = \frac{1}{(2\pi)^2} \sum_{k \neq 0, k \in \mathbb{Z}^2} \frac{\exp[ik \cdot (x - y)]}{k^2} \tag{1.5}$$

is the fundamental solution of the Poisson equation in T, which makes sense by virtue of the neutrality condition

$$\int \omega(x)\, dx = 0 \tag{1.6}$$

Equation (1.6) is a consequence of the periodicity of u and of the circulation theorem.

The initial value problem associated with (1.1) and (1.3) has been widely investigated and makes sense for initial data $\omega_0 \in L_\infty(T)$. Moreover, it is well known that any functional of the form

$$\beta H + \gamma E + \int_T \phi(\omega)\, dx \tag{1.7}$$

where β, $\gamma \in \mathbb{R}$ and $\phi \in C_0(\mathbb{R})$ and

$$H = \frac{1}{2} \int_T u^2(x)\, dx \qquad \text{(energy)} \tag{1.8}$$

$$E = \frac{1}{2} \int_T \omega^2(x)\, dx \qquad \text{(enstrophy)} \tag{1.9}$$

are first integrals for the Euler flow.

It is immediately seen that

$$\beta H + \gamma E = \tfrac{1}{2}(\omega, (\gamma \mathbf{1} - \beta \Delta^{-1})\, \omega); \qquad \beta, \gamma > 0 \tag{1.10}$$

is a quadratic form generating a Gaussian measure, which is formally defined as

$$\mu_{\beta,\gamma}(d\omega) = \prod_{k \neq 0, k \in \mathbb{Z}^2} \frac{\exp[-\frac{1}{2} |\hat{\omega}_k|^2 (\gamma + \beta/k^2)]}{2\pi(\gamma + \beta/k^2)^{-1}}\, d\hat{\omega}_k \tag{1.11}$$

where $\hat{\omega}_k$ are the Fourier coefficients of ω and $d\hat{\omega}_k = d \operatorname{Re} \hat{\omega}_k\, d \operatorname{Im} \hat{\omega}_k$.

As remarked in Refs. 3–11, the vector field associated with the Euler equation is divergenceless (in the space of the Fourier coefficients), so that the measure (1.11) is expected to be invariant for the Euler flow. Unfortunately, it is only "formally" invariant, because the set of all $\omega \in L_\infty(T)$ has $\mu_{\beta,\gamma}$-measure zero, so that the construction of a $\mu_{\beta,\gamma}$-almost everywhere defined Euler flow is problematic because $\mu_{\beta,\gamma}$ is concentrated on a set of distributions for which the initial value problem does not easily make sense. Nevertheless, one can construct a (possibly nonunique) one-parameter group of unitary operators acting on $L_2(d\mu_{\beta,\gamma})$, whose generator coincides with the Liouville operator associated with the Euler flow on a domain of sufficiently smooth functions.[7,8]

The problem of constructing invariant measures associated with the first integrals (1.7) reduces to that of giving sense to the measures

$$\frac{1}{\text{Norm}} \mu_{\beta,\gamma}(d\omega) \exp \int \phi(\omega)\, dx \tag{1.12}$$

This problem is analogous to that of Euclidean quantum field theory[9] with the free covariance $(-\Delta + m)^{-1}(x, y)$ replaced by $(\gamma - \beta\Delta^{-1})^{-1}(x, y)$.

In Section 3 we give heuristic arguments suggesting the impossibility of constructing non-Gaussian invariant measures of the form (1.12).

The Gibbs measures for the vortex model also can be considered as invariant measures for the Euler flow. The vortex model is defined through the system of ordinary differential equations

$$\dot{x}_i = \sum_{j \neq i} K(x_i, x_j)\, \alpha_j, \qquad \alpha_j = \pm e, \quad i = 1,\dots, N, \quad x_i \in T \tag{1.13}$$

The flow

$$\sum \alpha_j\, \delta(x - x_j) \to \sum \alpha_j\, \delta(x - x_j(t)) \tag{1.14}$$

can be interpreted as a generalized solution of the Euler flow.[10,11]

The vortex system (1.13) is Hamiltonian with the Hamilton function given by

$$H = \tfrac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j V(x_i, x_j) \tag{1.15}$$

and the conjugate variables are proportional to the coordinates of x_i, $i = 1,\dots, N$.

The canonical Gibbs measure for such a system

$$\frac{1}{\text{Norm}} (\exp -\beta H)\, dx_1 \cdots dx_N \tag{1.16}$$

has been proposed by Onsager.[12] Obviously the measure (1.16) coincides with the canonical measure associated with a two-dimensional classical Coulomb system. Such a system has been widely investigated from a rigorous point of view[13] and the existence of the measure (1.16) has been established for values $\beta < 4\pi/e^2$.

As pointed out by Kraichnan,[14] there is a close analogy between the measures (1.16) and (1.11). In the present paper we establish a rigorous connection between two such measures.

Before stating precisely our result, we mention that the vortex model (as a finite-dimensional dynamical system) is employed as a numerical algorithm to simulate the behavior of the (continuous) Euler flow (see Ref. 15 and also Refs. 11 and 16 for a more general, but weaker result).

The basic underlying idea in these approaches is the following. Let $\omega_0 \to \omega_t$ be a solution of Eqs. (1.1) and (1.3) associated with the initial profile ω_0. Approximate ω_0 by

$$\omega_0(x) \approx \sum \alpha_j \,\delta(x - x_j), \qquad j = 1,..., N \tag{1.17}$$

and evolve the rhs of (1.17) according to the flow (1.14). Then $\sum \alpha_j \,\delta(x - x_j(t))$ should be (and actually is) close to ω_t and the error vanishes when $N \to \infty$.

Actually this kind of result is obtained by replacing K by K_ε in Eq. (1.13), where $K_\varepsilon = \nabla^\perp V_\varepsilon$ and V_ε is a regularization of V smoothing the logarithmic divergence when $x \approx y$. This means that one considers finite blobs of vorticity (approximately of diameter ε) instead of point vortices. Then the limit $N \to \infty$ is performed simultaneously to the limit $\varepsilon \to 0$ with $\varepsilon \approx N^{-\delta}$, $0 < \delta < 1$. We finally notice that, due to a possible hyperbolicity of the motion, the error at time t can be hardly estimated better than $N^{-p} C(p) \exp Ct$, with p arbitrarily large, $C(p)$ diverging with p.

Here we prove that a similar limit holds in the framework of statistical mechanics. Namely, the Gibbs measure associated with a gas of vortex blobs interacting via a two-body interaction $V_\varepsilon(x, y)$, converges, in the limit $N \to \infty$, $Ne^2 \to \text{const}$, $\varepsilon \to 0$ suitably, to the Gaussian measure (1.11). This means that a long-time control of the Euler flow by means of the vortex dynamics, if lost in terms of individual solutions, can be recovered in statistical terms, provided that an equilibrium of the form (1.11) is achieved.

In the next section we establish and prove our main result. The last section is devoted to concluding remarks.

2. RESULTS AND PROOFS

We consider a gas of vortices interacting via a two-body interaction defined as

$$V_\varepsilon(x, y) = \frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{e^{ik(x-y)} e^{-\varepsilon k^2}}{k^2} \tag{2.1}$$

The canonical Gibbs measures $\mu^N_{\beta,\sigma}$, thought of as defined on the fields of the form $\omega(dx) = \sum_{j=1,N} \alpha_j \delta(x - x_j)\, dx$, have a characteristic function given by

$$\begin{aligned}\int \{\exp[i\omega(f)]\}\, \mu^N_{\beta,\sigma}(d\omega) = (Z^N_{\beta,\sigma})^{-1} & \sum_{\substack{\alpha_1 \cdots \alpha_N \\ \alpha_i = \pm\sqrt{\sigma}, \sum \alpha_i = 0}} \int dx_1 \cdots dx_N \\ & \times \exp\left[-\beta/2 \sum_{i \neq j} V_\varepsilon(x_i, x_j)\, \alpha_i \alpha_j\right] \\ & \times \exp\left[i \sum_{j=1,N} \alpha_j f(x_j)\right] \end{aligned} \tag{2.2}$$

where $\omega(f) = \sum_{j=1,N} \alpha_j f(x_j)$, f is any continuous real function, $Z^N_{\beta,\sigma}$ is the partition function, and $\sigma > 0$.

We define the space

$$\Omega = \prod_{k \in \mathbb{Z}^2} \Omega_k, \qquad \Omega_k = \mathbb{C} \tag{2.3}$$

of the Fourier coefficients of the fields ω equipped with the product topology. Then $\mu^N_{\beta,\sigma}$ can be interpreted as a family of Borel probability measures on Ω.

We find it convenient to rewrite the characteristic functionals (2.2) as

$$\begin{aligned}\mu^N_{\beta,\sigma}(\exp i\omega(f)) = (Z^N_{\beta,\sigma})^{-1} & \int \frac{dx_1 \cdots dx_N}{(2\pi)^{2N}} \int d\nu(\alpha_1) \cdots d\nu(\alpha_N) \\ & \times \mathbb{1}\left(\sum \alpha_j = 0\right) \exp\left[-\beta/2 \sum_{i \neq j} \alpha_i \alpha_j V_\varepsilon(x_i, x_j) \right. \\ & \left. + i \sum_{j=1,N} \alpha_j f(x_j)\right] \end{aligned} \tag{2.4}$$

where

$$\nu(d\alpha) = \tfrac{1}{2}(\delta_{\sqrt{\sigma}} + \delta_{-\sqrt{\sigma}})(d\alpha) \tag{2.5}$$

and $\mathbb{1}$ $(\sum \alpha_j = 0)$ is the indicator of the set $\{\omega \mid \sum \alpha_j = 0\}$.

Furthermore, $\mu_{\beta,\gamma}$ is a Borel probability measure on Ω and

$$\mu_{\beta,\gamma}(\exp i\omega(f)) = \exp\left(-\frac{1}{2}\sum_{k\neq 0} |\hat{f}_k|^2 \frac{k^2}{k^2\gamma+\beta}\right) \tag{2.6}$$

The main result of this paper is the following.

Theorem. The sequence $\mu^N_{\beta,\sigma}$ converges weakly in the limit $N\to\infty$ to $\mu_{\beta,\gamma}$ if $\sigma=(2\pi)^2/\gamma N$, $\varepsilon=N^{-\delta}$, for some $\delta\in(0,\gamma/2\pi\beta)$.

Proof. First we prove the convergence of the corresponding characteristic functionals.

Let $\phi(x)$ be the Gaussian random field with covariance given by βV_ε. Then, denoting by $\mathbf{E}_\varepsilon$ the expectation with respect to the distribution of ϕ, we have

$$\exp\left[-\beta/2\sum_{i,j=1}^N \alpha_i\alpha_j V_\varepsilon(x_i,x_j)\right] = \mathbf{E}_\varepsilon\left[\exp i\sum_{j=1}^N \alpha_j\phi(x_j)\right] \tag{2.7}$$

Therefore, for $f\in C_0(T)$, the characteristic functional of $\mu^N_{\beta,\sigma}$ can be written as

$$\mu^N_{\beta,\sigma}(\exp i\omega(f)) = (\bar{Z}_{\beta,\sigma})^{-1}\,\mathbf{E}_\varepsilon\left\{\int\frac{dx_1\cdots dx_N}{(2\pi)^{2N}}\int d\nu^*_N(\alpha_1,\ldots,\alpha_N)\right.$$
$$\left.\times\exp\left[i\sum_{j=1,N}\alpha_j P(f+\phi)(x_j)\right]\right\} \tag{2.8}$$

where $d\nu^*_N(\alpha_1,\ldots,\alpha_N)$ is the normalized probability measure on $\{-\sqrt{\sigma}, +\sqrt{\sigma}\}^N$ with the constraint $\sum_{i=1,N}\alpha_i=0$ (which imposes that N be an even integer),

$$\bar{Z}_{\beta,\sigma} = \mathbf{E}_\varepsilon\left\{\int\frac{dx_1\cdots dx_N}{(2\pi)^{2N}}\int d\nu^*_N(\alpha_1,\ldots,\alpha_N)\exp\left[i\sum_{j=1,N}P\phi(x_j)\,\alpha_j\right]\right\} \tag{2.9}$$

and

$$Pf(x) = f(x) - \int_T f(x)\,dx/(2\pi)^2 \tag{2.10}$$

Since

$$\int dx/(2\pi)^2\exp[i\alpha_j P(f+\phi)(x_j)] = \exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right] + O(1/N^{3/2})$$

we extract the leading term in the rhs of Eq. (2.8) by using the expansion

$$\begin{aligned}\exp\left[i\sum_{j=1,N}\alpha_j P(f+\phi)(x_j)\right]&=\exp\left[-\frac{1}{2\gamma}\|P(f+\phi)\|_2^2\right]\\&\quad+\sum_{k=1}^{N}\left(\prod_{j=1}^{k-1}\exp[i\alpha_j P(f+\phi)(x_j)]\right.\\&\quad\times\left\{\exp[i\alpha_k P(f+\phi)(x_k)]-\exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right]\right\}\\&\quad\left.\times\prod_{j=k+1}^{N}\exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right]\right)\end{aligned}\tag{2.11}$$

We get

$$\mu^N_{\beta,\sigma}(\exp i\omega(f))\,\bar Z_{\beta,\sigma}=\mathbf{E}_\varepsilon\left\{\exp\left[-\frac{1}{2\gamma}\|P(f+\phi)\|_2^2\right]\right\}+A(f,N)\tag{2.12}$$

where

$$A(f,N)=\mathbf{E}_\varepsilon\left\{\int\frac{dx_1\cdots dx_N}{(2\pi)^{2N}}\int dv^*_N(\alpha_1,\ldots,\alpha_N)\,\psi_f(N)\right\}\tag{2.13}$$

and $\psi_f(N)$ is the sum on the rhs of Eq. (2.11).

By Eq. (2.12) we get

$$\begin{aligned}\mu^N_{\beta,\sigma}(\exp i\omega(f))&=\frac{\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(f+\phi)\|_2^2]\}}{\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(\phi)\|_2^2]\}}\\&\quad\times\frac{1}{1+A(0,N)/\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(\phi)\|_2^2]\}}\\&\quad+\frac{A(f,N)/\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(\phi)\|_2^2]\}}{1+A(0,N)/\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(\phi)\|_2^2]\}}\end{aligned}\tag{2.14}$$

By means of explicit calculations of Gaussian integrals we obtain

$$\frac{\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(f+\phi)\|_2^2]\}}{\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(\phi)\|_2^2]\}}=\exp\left[-\tfrac{1}{2}\sum_{k\neq 0}|\hat f_k|^2\frac{(\exp\varepsilon k^2)\,k^2}{\gamma k^2\exp(\varepsilon k^2+\beta)}\right]\tag{2.15}$$

Therefore we prove the theorem once we prove that, for all f,

$$\frac{A(f,N)}{\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\|P(\phi)\|_2^2]\}}\tag{2.16}$$

vanishes in the limit $N\to\infty$, $\sigma\to 0$, $\sigma=(2\pi)^2/\gamma N$, $\varepsilon=N^{-\delta}$, with $\delta\in(0,\gamma/2\beta\pi)$.

We have

$$\mathbf{E}_\varepsilon\left\{\exp\left[-\frac{1}{2\gamma}\|P(\phi)\|_2^2\right]\right\}$$
$$= \exp\left\{-\frac{1}{2}\sum_{k\neq 0}\log\left(1+\frac{\beta\exp(-\varepsilon k^2)}{\gamma k^2}\right)\right.$$
$$> \exp\left(-\frac{\beta}{2\gamma}\sum_{k\neq 0}\frac{\exp(-\varepsilon k^2)}{k^2}\right) > \text{const}\times\varepsilon^{\pi\beta/\gamma} \tag{2.17}$$

Let us call $\psi_k(N)$ a generic term in $\psi_f(N)$. Performing the scaling $\alpha_j \to \alpha_j/(\gamma N)^{1/2}$ simultaneously on all variables $(\alpha_1,..., \alpha_N)$, we obtain, denoting by $d\tilde{v}_N(\alpha_1,..., \alpha_N)$ the transformed measure,

$$\mathbf{E}_\varepsilon\left\{\int\frac{dx_1\cdots dx_N}{(2\pi)^{2N}}\int dv_N^*(\alpha_1,..., \alpha_N)\,\psi_k(N)\right\}$$
$$= \mathbf{E}_\varepsilon\left(\int\frac{dx_1\cdots dx_N}{(2\pi)^{2N}}\int d\tilde{v}_N^*(\alpha_1,..., \alpha_N)\right.$$
$$\times\prod_{j=1}^{k-1}\exp[i\alpha_j/(\gamma N)^{1/2}\,P(f+\phi)(x_j)]$$
$$\times\left\{\exp[i\alpha_k/(\gamma N)^{1/2}\,P(f+\phi)(x_k)]-\exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right]\right\}$$
$$\left.\times\prod_{j=k+1}^{N}\exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right]\right) \tag{2.18}$$

By definition of $P(f+\phi)$,

$$\int dx_k\, i\alpha_k/(\gamma N)^{1/2}\,P(f+\phi)(x_k)=0 \tag{2.19}$$

On the other hand, using $\alpha_j^2=1$, we have

$$\int dx_k\,\alpha_k^2/(2\gamma N)\,P(f+\phi)(x_k)^2=\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\,(2\pi)^2 \tag{2.20}$$

Therefore the rhs of Eq. (2.18) is equal to

$$\mathbf{E}_\varepsilon\left[\int\frac{dx_1\cdots dx_N}{(2\pi)^{2N}}\int d\tilde{v}_N^*(\alpha_1,..., \alpha_N)\prod_{j=1}^{k-1}\exp[i\alpha_j/(\gamma N)^{1/2}\,P(f+\phi)(x_j)]\right.$$
$$\times(\{\exp[i\alpha_k/(\gamma N)^{1/2}\,P(f+\phi)(x_k)]-1-i\alpha_k/(\gamma N)^{1/2}\,P(f+\phi)(x_k)$$
$$+\alpha^2/2\gamma N[P(f+\phi)(x_k)]^2\}$$
$$\left.-\left\{\exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right]-1+\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right\}\right)$$
$$\left.\times\prod_{j=k+1}^{N}\exp\left[-\frac{1}{2\gamma N}\|P(f+\phi)\|_2^2\right]\right] \tag{2.21}$$

Therefore by the Taylor formula we get

$$\left| \mathbf{E}_\varepsilon \left\{ \int \frac{dx_1 \cdots dx_N}{(2\pi)^{2N}} \int dv_N^*(\alpha_1, ..., \alpha_N)\, \psi_k(N) \right\} \right|$$
$$\leqslant \frac{1}{3!\,(\gamma N)^{3/2}} \mathbf{E}_\varepsilon \left\{ \int \frac{dx_k}{(2\pi)^2} |P(f+\phi)(x_k)|^3 \right\}$$
$$+ \frac{1}{8(\gamma N)^2} \mathbf{E}_\varepsilon \{ \|P(f+\phi)\|_2^4 \} \tag{2.22}$$

Using the Holder inequality, we get that the rhs of Eq. (2.22) does not exceed

$$\frac{1}{3!\,(\gamma N)^{3/2}} [\mathbf{E}_\varepsilon\{ \|P(f+\phi)\|_4^4 \}]^{3/4} + \frac{1}{8(\gamma N)^2} \mathbf{E}_\varepsilon\{ \|P(f+\phi)\|_4^4 \} \tag{2.23}$$

Thus,

$$|A(f, N)| \leqslant \text{const} \times N/(N\gamma)^{3/2}\, [V_\varepsilon(0,0)^2 + \|f\|_4^4] \tag{2.24}$$

Furthermore,

$$V_\varepsilon(0,0) \leqslant -\text{const} \times \log \varepsilon = \text{const} \times \delta \log N \tag{2.25}$$

Combining (2.24), (2.25), and (2.17), we obtain

$$\frac{A(f, N)}{\mathbf{E}_\varepsilon\{\exp[-(1/2\gamma)\, \|P(\phi)\|_2^2]\}} \leqslant \text{const} \frac{(\log N)^2}{N^{1/2}} N^{\delta\beta\pi/\gamma} \tag{2.26}$$

It remains to show that the convergence of the characteristic functionals implies weak convergence. We observe that Ω is metrizable with metric function given, for instance, by

$$d(\omega, \omega') = \sum_{k \neq 0} e^{-k^2} \min\{1, |\hat{\omega}_k - \hat{\omega}'_k|\} \tag{2.27}$$

Furthermore, the set of all continuous bounded cylindrical functions, i.e., the functions depending only on a finite number of $\hat{\omega}_k$, is dense, with respect to the uniform topology, in the set of all continuous bounded functions. Therefore we need to prove only the weak convergence, for any finite $\Lambda \subset \mathbb{Z}^2$, of $\mu_{\beta,\sigma}^N | \Omega_\Lambda$ to $\mu_{\beta,\gamma} | \Omega_\Lambda$, where $\Omega_\Lambda = \prod_{k \in \Lambda} \Omega_k$. This immediately follows since, in finite-dimensional spaces, the convergence of the characteristic functions to the characteristic function of a given measure implies weak convergence.

3. CONCLUDING REMARKS

It is obviously true that the results of this paper are valid for more general situations than those explicitly treated. For example, more general measures ν for which $\nu(\alpha^2)=(2\pi)^2/\gamma N$ and $\nu(\alpha^4)=\text{const}\times N^{-2}$, different cutoffs, or more general bounded domains (in which the Fourier coefficients are replaced by the projections on the eigenfunctions of the Laplace operators[6,8]) are allowed.

Furthermore, the theorem in Section 2 can be proved by replacing $\mu^N_{\beta,\sigma}$ by $\mu^z_{\beta,\sigma}$, the grand canonical Gibbs measure with activity z. In this case the conditions are $z\to\infty$, $\sigma\to 0$, $z\sigma=(2\pi)^2/\gamma$. We omit the details.

The results of the above section show that, among the measures generated by the first integrals (1.7), the Gaussian ones play a special role. An additional comment to this is the following. The Gaussian measures generated by a linear combination of the energy and the enstrophy enjoy the property of being invariant with respect to the finite-dimensional dynamics obtained by projecting the Euler equation on a finite, symmetric subset of the k space $\mathbb{Z}^2$. Other non-Gaussian measures associated with a $\phi\neq 0$ (if they exist) cannot have this feature. Considering that the small wave numbers are made small by the viscosity, this property seems to be relevant from a physical point of view. In any case, heuristic arguments based on the renormalization group analysis of the measures (1.2) strongly suggest the impossibility of perturbing the measure $\mu_{\beta,\sigma}$ by a nonquadratic potential.

Notice that the covariance of a "free" field ω distributed via $\mu_{1,1}$ has covariance whose Fourier transform is

$$\tilde{C}(k)=\frac{k^2}{1+k^2}=\lim_{N\to\infty}\sum_{n=0}^{N-1}\frac{k^2}{1+k^2}\left[\chi(2^{-n-1}k)-\chi(2^{-n}k)\right]+\frac{k^2\chi(k)}{1+k^2} \tag{3.1}$$

where $\chi(k)$ is a C^∞ function of compact support, such that $\chi(0)=1$.

Then the "cutoff" field $\omega^{(N)}(x)$ of covariance $C^{(N)}(x)$ can be expanded as a sum of independent, smooth Gaussian fields, whose covariances are given by the different addends on the rhs of (3.1):

$$\omega^{(N)}(x)=\sum_{n=0}^{N-1}\omega_n(x)+\bar{\omega}(x) \tag{3.2}$$

Let us notice that the covariances $C_n(x)$ of $\omega_n(x)$ satisfy the scaling relation ($d=2$):

$$C_n(x)=2^{nd}\bar{C}_n(2^n x) \tag{3.3}$$

where

$$\bar{C}_n(x) = (2\pi)^{-2} \sum_{k \neq 0} \frac{k^2}{2^{-2n} + k^2} [\chi(2^{-1}k) - \chi(k)]\, e^{ikx} \tag{3.4}$$

is essentially independent of n.

The usual way of giving a meaning to the measure (1.12) is to study the limit (see Ref. 17 for a recent review on the subject)

$$\lim_{N \to \infty} \int P(d\omega^{(N)}) \exp\left[\int \phi(\omega^{(N)})\, dx\right] \tag{3.5}$$

By (3.3) and (3.4) the properties of the above limit should not change in an essential way if the field $\omega^{(N)}(x)$ is substituted by its "hierarchical" approximation

$$\varphi^{(N)}(x) = \sum_{n=0}^{N-1} 2^n z_{\Delta(x,n)} \tag{3.6}$$

where $\Delta(x, n)$ is the tessera containing x, belonging to a fixed pavement Q_n of $\mathbf{T}_2$, consisting of squares of side $2^{-n}(2\pi)$. The variables z_Δ are independent normal variables.

Let us define normalized variables

$$\psi_\Delta^{(N)} = \varphi^{(N)}(x) \Big/ \left(\sum_{n=0}^{N-1} 2^{2n}\right)^{1/2} \tag{3.7}$$

where $\Delta \in Q_N$ and x is any point in Δ. Equations (3.6) and (3.7) imply that

$$\psi_\Delta^{(N)} \approx \frac{\sqrt{3}}{2} z_\Delta + \tfrac{1}{2}\psi_{\tilde{\Delta}}^{(N-1)} \tag{3.8}$$

where $\Delta \in Q_N$, $\tilde{\Delta} \in Q_{N-1}$, and $\tilde{\Delta} \supset \Delta$.

Then we have

$$\begin{aligned} &\int P(d\varphi^{(N)}) \exp\left[\int \Phi(\varphi^{(N)})\, dx\right] \\ &= \int \prod_{\Delta \in Q_N} P(d\psi_\Delta^{(N)}) \exp[V(\psi_\Delta^{(N)})] \\ &= \int \prod_{\Delta \in Q_{N-1}} P(d\psi_\Delta^{(N-1)}) \exp[TV(\psi_\Delta^{(N-1)})] \\ &= \int \prod_{\Delta \in Q_{N-k}} P(d\psi_\Delta^{(N-k)}) \exp[T^k V(\psi_\Delta^{(N-k)})] \end{aligned} \tag{3.9}$$

which defines the "effective potential" $T^k V$ on the scale 2^{-k} and the renormalization group transformation T. We have, by (3.8),

$$TV(\psi) = 4 \log \int \exp[V(\sqrt{3}/2z + \tfrac{1}{2}\psi)] \exp(-z^2/2)\, dz/(2\pi)^{1/2} \quad (3.10)$$

If $V(x) = \lambda^{(N)} H_{2n}(x)$ [where $H_{2n}(x)$ is the Hermite polynomial of order n], we have, at the first order in λ,

$$TV(\psi) = \frac{4}{2^{2N}} \lambda^{(N)} H_{2n}(\psi) \quad (3.11)$$

This suggest that the contribution of the nonquadratic part of $T^k V(x)$ goes to zero as $k \to \infty$ if $\lambda^{(N)}$ is chosen small enough (independent of the cutoff N). This situation is analogous to that of the Φ_d^4 quantum field theory with $d > 4$, where it is generally believed that it is impossible to obtain a nontrivial limit when $N \to \infty$.[18]

Finally, we discuss the possibility of obtaining the results of the theorem in Section 2 without making use of any regularization of $V(x, y)$.

Since $[\tilde{\gamma} = \gamma/(2\pi)^2]$

$$\int \{\exp[i\omega(f)]\}\, \mu_{\beta,\sigma}^N(d\omega) = \int \left\{\exp\left[i\omega\left(\frac{f}{(\gamma N)^{1/2}}\right)\right]\right\} \mu_{\beta/\tilde{\gamma}N,1}^N(d\omega) \quad (3.12)$$

for any N large enough (so that $\beta/\tilde{\gamma}N < 4\pi$; see Ref. 13), the limit $\varepsilon \to 0$ is meaningful. Then one could hope that it is possible to control the limit $N \to \infty$ uniformly in ε by using more powerful techniques for the estimation of the two quantities in Eq. (2.16). However, this does not seem to be the case.

Let us consider, for example, the first term in Eq. (2.16). We can write

$$\begin{aligned} &A(0, N)/\mathbf{E}_\varepsilon\{\exp(-1/2\gamma\, \|P\Phi\|_2^2)\} \\ &\quad = \bar{A}(0, N)/\mathbf{E}_\varepsilon\{\exp((\beta/2\gamma)\, V_\varepsilon(0,0) - 1/2\gamma\, \|P\Phi\|_2^2)\} \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} \bar{A}(0, N) = \mathbf{E}_\varepsilon \Bigg(&\int dx_1 \cdots dx_N/(2\pi)^{2N} \int dv_N^*(\alpha_1,\dots,\alpha_N) \\ &\times \sum_{k=1}^{N} \left[\prod_{j=1}^{k-1} {:}\exp(i\alpha_j P(\Phi)(x_j)){:}\right] \{{:}\exp(i\alpha_k P(\Phi)(x_k)){:} \\ &- \exp[(\beta/2\tilde{\gamma}N)\, V_\varepsilon(0,0) - (\tfrac{1}{2}\gamma N)\, \|P(\Phi)\|_2^2]\} \\ &\times \left\{ \prod_{j=k+1}^{N} \exp[(\beta/2\tilde{\gamma}N)\, V_\varepsilon(0,0) - (\tfrac{1}{2}\gamma N)\, \|P(\Phi)\|_2^2] \right\} \Bigg) \end{aligned} \quad (3.14)$$

:-: being the Wick product.

By (2.17) the limit for $\varepsilon \to 0$ of the denominator on the rhs of (3.13) exists and is greater than zero. Then one is faced with the problem of showing that $|\bar{A}(0, N)| \to 0$ for $N \to \infty$, uniformly in ε. It is clear that, if this is true, one should also be able to prove that, uniformly in ε,

$$Z_{\beta, 1/\tilde{\gamma} N} = \mathbf{E}_\varepsilon \left\{ \int dx_1 \cdots dx_N/(2\pi)^{2N} \int dv_N^*(\alpha_1, ..., \alpha_N) \right.$$
$$\left. \times \prod_{j=1}^{N} :\exp(i\alpha_j P(\Phi)(x_j)): \right\} \leqslant c(N) \tag{3.15}$$

with $c(N)$ a slowly growing function of N.

One can easily show that the bound (3.15) is true iff the analogous bound is true for a Yukawa gas of particles of charges $\pm 1/(\tilde{\gamma} N)^{1/2}$ in a fixed volume Λ with $|\Lambda| = 1$. Then one can try to use the results of Ref. 19 through the obvious identity

$$\tilde{Z}_{\beta, \sigma} = d^N/d\lambda^N \Xi_{\beta, \sigma}(\lambda)\,|_{\lambda = 0} \tag{3.16}$$

where $\tilde{Z}_{\beta, \sigma}$ and $\Xi_{\beta, \sigma}(\lambda)$ are, respectively, the canonical and the grand canonical partition functions of the Yukawa gas.

In Ref. 19 it is shown that

$$\log \Xi_{\beta, \sigma}(\lambda) \leqslant |\lambda| + \sum_{n=2}^{\infty} (c\,|\lambda|)^n\,(\beta\sigma)^{n-1} \tag{3.17}$$

for a suitable constant c. Then, by the Cauchy formula

$$\tilde{Z}_{\beta, \sigma} \leqslant N! \inf_R \exp\left[R + cR \sum_{n=1}^{\infty} (c\beta\sigma R)^n \right] \Big/ R^N \tag{3.18}$$

which implies

$$\begin{aligned} \tilde{Z}_{\beta, 1/\tilde{\gamma} N} &\leqslant N! \inf \exp\left[R + cR \sum (c\beta R/\tilde{\gamma} N)^n \right] \Big/ R^N \\ &\approx \bar{c}^N (N!/N^N)\, e^N \approx \bar{c}^N \sqrt{N} \end{aligned} \tag{3.19}$$

for a suitable $\bar{c} > 1$.

The bound of $\tilde{Z}_{\beta, \sigma}$ obtained in Ref. 13 leads to an even worse estimate.

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